Clareus Scientific Science and Engineering
Volume 2 Issue 8 October 2025
DOI: 10.70012/CSSE.02.049
ISSN: 3065-1182

# On the Onset of Convection in a Hele-Shaw Geometry under the Influence of Mechanical Vibrations

*Citation:* Yazdan Pedram Razi., et al. "On the Onset of Convection in a Hele-Shaw Geometry under the Influence of Mechanical Vibrations". Clareus Scientific Science and Engineering 2.8 (2025): 02-07.

*Article Type:* Research Article

*Received:* September 6, 2025

*Published:* September 25, 2025



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## Abstract

The onset of natural convection in a vertical Hele-Shaw configuration under the action of mechanical vibration is studied. Vibration is considered to be a harmonic one and its direction is considered parallel to the temperature gradient. Linear stability analysis for small *Pr* number leads us to the Mathieu equation. The so-called time averaged method is applied to this equation which gives us a closed form relation for the critical $Ra_T$ number for the harmonic (synchronous) solution. The results are in a perfect agreement with published numerical results.

## Introduction

Mechanical vibration, due to its role in modifying the driving force in natural convection, has been the subject of numerous studies. The aim of this paper is to investigate analytically the influence of mechanical vibration on the stability threshold of the conductive solution in a vertical Hele-Shaw configuration. The direction of vibration was parallel to the temperature gradient. It was shown recently in a study by Aniss et al [1]. that, for small values of Pr number, the stability analysis of convective flow in the Hele-Shaw cell under the effect of mechanical vibration leads to the study of the Mathieu equation. The stability diagrams for this problem were obtained using Hill's infinite determinant. In this work we show, how by applying the time-averaged method to the Mathieu equation, we may obtain an analytical relation for the critical Rayleigh number as a function of vibrational parameters. Our objective is to obtain some analytical relations to shed some light on the influence of thermo-vibrational parameters on the convective stability of the problem studied by Aniss et al [1].

## Linear Stability Analysis

We considered a vertical Hele-Shaw geometry having infinite extension in the x direction, Figure 1. The two horizontal plates were kept at two constant but different temperatures while the lateral walls were thermally insulated; the plates were rigid and impermeable. This configuration was subjected to harmonic mechanical vibrations. As indicated in [1], by performing the linear stability analysis, the governing equations for low Prandtl number fluids may be transformed into a damped Mathieu equation:

$$\frac{d^2 f}{dt^2} + A\frac{df}{dt} + (B + C\sin\Omega t)\, f = 0,$$
$$A = \pi^2 + k^2 + 12\,Pr^*, \quad B = 12\,Pr^*(k^2+\pi^2) - Ra_T\, Pr^* \frac{k^2}{k^2+\pi^2},$$
$$C = Ra_T\, Pr^*\, Fr\Omega^2 \frac{k^2}{k^2+\pi^2} \qquad (1)$$

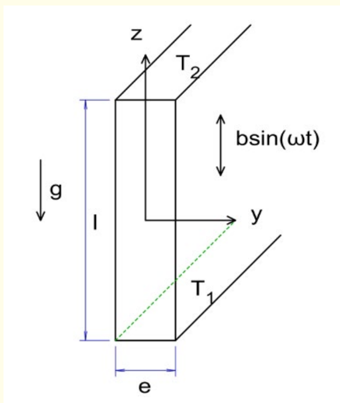


***Figure 1:*** The Hele-Shaw geometry.

where $Pr^* = Pr/\varepsilon^2$, $Pr^* = O(1)$ represents the modified Prandtl number, $Fr = \alpha^2 b/gH^4$ represents Froude number and $Ra_T = g\beta\Delta THe^2/\nu\alpha$ is the thermal Rayleigh number. $\Omega$ is non-dimensional frequency, $b$ is the amplitude of vibration, $H$ is the height of the cell, $k$ is the wave number, $g$ is the acceleration of gravity, $\alpha$ is the thermal diffusivity of the fluid and $\varepsilon$ represents the aspect ratio of the cell $\varepsilon = e/H << 1$ (e is the distance between the vertical plates).

***Time-averaged formulation***

In order to study the mean behavior of Eq.(1), the time-averaged method was used under the limiting conditions of high-frequency and small-amplitude of vibration. Later, we clarify what is meant by high-frequency and small amplitude. Under these conditions the solution of Eq. (1) can be expressed as the superposition of two fields having two different time scales [2, 3]; $f_0$ is the slow field where the characteristic time is much larger than the vibration period while, for the fast field $\eta$, the characteristic time is the same as vibration period. So we may write:

$$f = f_0(t) + \eta(\Omega t) \qquad (2)$$

In the above transformation, the fast field may be thought of as a small correction to the slow field ($\eta << f_0$). On substituting Eq.(2) into Eq.(1), and averaging over a vibration period, two coupled systems of equations are distinguished:

$$\frac{d^2 f_0}{dt^2}+(\pi^2+k^2+12\,Pr^*)\frac{df_0}{dt}+\left[12\,Pr^*(k^2+\pi^2)-Ra_T\,Pr^*\frac{k^2}{k^2+\pi^2}\right]f_0$$
$$-Ra_T\,Pr^*\frac{k^2}{k^2+\pi^2}Fr\,\Omega^2\,\overline{\eta\sin\Omega t}=0, \tag{3}$$

Eq.(3) governs the field having slow variation with respect to time while Eq.(4) governs the fields having fast variation with respect to time.

$$\frac{d^2\eta}{dt^2}+(\pi^2+k^2+12\,Pr^*)\frac{d\eta}{dt}-Ra_T\,Pr^*\frac{k^2}{k^2+\pi^2}Fr\,\Omega^2\sin\Omega t\,f_0$$
$$-\left[12\,Pr^*(k^2+\pi^2)-Ra_T\,Pr^*\frac{k^2}{k^2+\pi^2}\right]\eta+Ra_T\,Pr^*\frac{k^2}{k^2+\pi^2}Fr\,\Omega^2\sin\Omega t\,\eta$$
$$-Ra_T\,Pr^*\frac{k^2}{k^2+\pi^2}Fr\,\Omega^2\,\overline{\eta\sin\Omega t}=0, \tag{4}$$

Here for a given function $f$, the average is defined as $\overline{f}=\frac{1}{\tau_v}\int_0^{\tau_v} f\,ds$ ($\tau_v$ is vibration period).

***Analytical relation***

In order to solve Eq.(3), a relation between slow and fast fields should be found. We write Eq. (4) in scaled form and we make the assumption that $d/dt \approx \Omega$ which allows us to compare the order magnitude of each term in Eq.(4). In doing this and with the assumptions that vibration has small amplitude ($Ra_T\,Fr\,Pr^* \ll 1$ which means that the amplitude of the fast field should be much smaller than the slow field) and high-frequency ($\Omega \gg (Pr^*, Ra_T\,Pr^*)$) which means that vibration period should be smaller than the thermal and hydrodynamic time scales and the thermal characteristic of the non-modulated systems respectively) allow us to to neglect the effect of small driving and diffusive terms in Eq. (4). So we get:

$$\eta=-FrRa_T\,Pr^*\,f_0\frac{k^2}{k^2+\pi^2}\sin\Omega t\quad, \tag{5}$$

It should be noted that the frequency of the fast oscillating field is $\Omega$. By replacing Eq.(5) in Eq.(3), we obtain the following relation for the marginal stability:

$$\frac{1}{2}Ra_T^2\,Pr^*\,Fr^2\,\Omega^2\left(\frac{k^2}{k^2+\pi^2}\right)^2-Ra_T\frac{k^2}{k^2+\pi^2}+12(k^2+\pi^2)=0, \tag{6}$$

It is obvious that the thermal Rayleigh number is related to thermal and vibrational ($Pr^*$, $\Omega$, $Fr$) parameters and the pattern selection parameter ($k$).

## Results and Discussion

In this section, different physical situations are considered namely in the absence of vibration, in the case of micro-gravity conditions and in the case of the combined effects (gravity and vibration).

***Convection in Hele-Shaw cell***

In order to validate our analytical relation, we study the onset of convection with no vibration. When there is no vibrational effect, we find, for the case of the layer heated from below, the classical result of $Ra_{Tc} = 48\pi^2$ and $k_c = \pi$. Further, we observe that, for the case of heating from above, there is no possibility of convective motion.

***Thermo-vibrational convection in micro-gravity***

Under this condition, as gravitational acceleration is zero in Eq.(6), we replace $Pr^*(Ra_T Fr\ \Omega)^2/2$ by $Ra_v$ which is the vibrational Rayleigh number ($Fr$ and $Ra_T$ both depend on gravitational acceleration but their product is independent of g). $Ra_v$ is always positive, so we conclude that, under microgravity conditions, the convective regime for harmonic mode (with frequency $\Omega$) does not exist.

***Thermo-vibrational convection in the presence of vibrational and gravitational acceleration fields***

Under this condition we can find, from Eq.(6), a simultaneous system of equations which relates the minimum Rayleigh number, the critical wave number and vibrational parameters as follows:

$$\begin{aligned} &\frac{\pi^2(\pi^2-k_c^2)}{(\pi^2+k_c^2)(2\pi^2-k_c^2)} = 6\,Pr^*\,Fr^2\,\Omega \\ &Ra_{Tc} = \frac{12(k_c^2+\pi^2)^2}{\pi^2 k_c^2}(2\pi^2-k_c^2) \end{aligned} \tag{7}$$

By using system (7), we obtain the influence of vibration on the critical parameters of the problem. Although it is possible to use the values given for bifurcation diagram $Ra_{Tc}$-$\Omega$ and $k_c$-$\Omega$ from figures presented in [1], in order to increase the precision of comparison we decided to perform the numerical procedure in detail there. The results are presented in table 1.

| ***Ω*** | ***Numerical Method*** | | ***Time-Averaged Method*** | |
|---|---|---|---|---|
| | $\boldsymbol{Ra_{Tc}}$ | $\boldsymbol{k_c}$ | $\boldsymbol{Ra_{Tc}}$ | $\boldsymbol{k_c}$ |
| 100 | 478.9378 | 3.1263 | 479.4708 | 3.1225 |
| 200 | 497.5964 | 3.0648 | 498.2403 | 3.0604 |
| 300 | 535.4260 | 2.9406 | 536.2421 | 2.9357 |
| 400 | 610.7706 | 2.7038 | 611.9828 | 2.6981 |
| 500 | 796.3512 | 2.2290 | 798.8769 | 2.2226 |

***Table 1:*** Comparison of stability analysis results for the critical Rayleigh and wave numbers from numerical and analytical methods for $Pr^* = 1$ and Fr $= 10^{-4}$.

These results are illustrated in Figure 2 and 3 too.

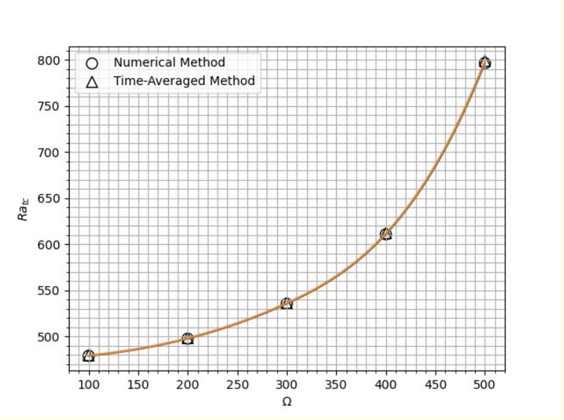


***Figure 2:*** The critical Rayleigh number as a function of non-dimensional frequency of vibrations for $Pr^* = 1$ and $Fr = 10^{-4}$.

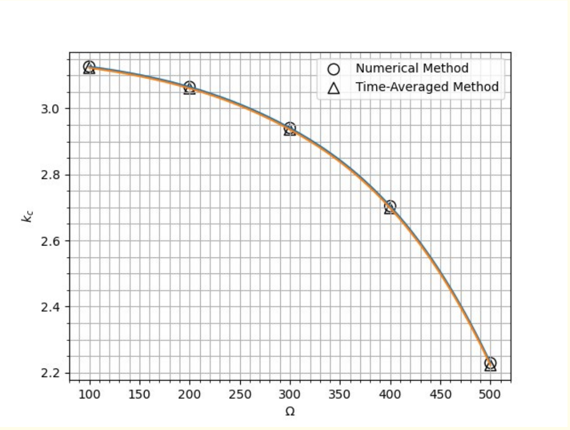


***Figure 3:*** The critical wave number as a function of non-dimensional frequency of vibrations for $Pr^* = 1$ and $Fr = 10^{-4}$.

***The Pattern formation aspects of vibrations***

Here the effect of vibrations on the convective patterns are qualitatively shown [4]:

As can be seen from the Figure 4 and 5, vibrations reduces the number of convective rolls in the limiting case of high frequency and small amplitude of vibrations.

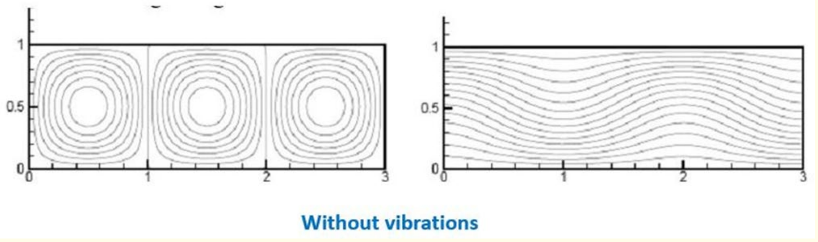


***Figure 4:*** The case of no vibrations.

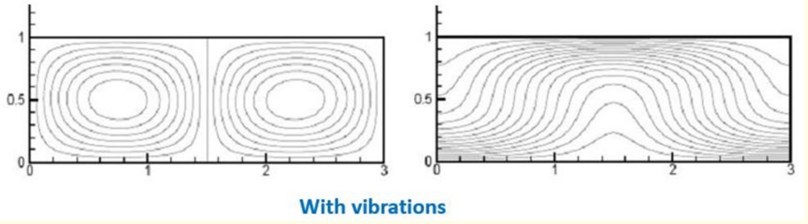


***Figure 5:*** The case of vibrations present.

## Conclusion

In this paper, the linear stability analysis of convective motion in a vertical Hele-Shaw configuration has been performed. The stability analysis at low Prandtl leads us to a Mathieu equation. It is shown that, under the limiting conditions of high frequency and small amplitude of vibration, the time-averaged method can be effectively adopted. The assumptions necessary for applying the time-averaged method have been obtained. Analytical relations are obtained from which we can calculate the critical thermal Rayleigh and wave numbers for given fluid properties and vibrational parameters. The results show that for fixed amplitude, the increase in vibration frequency increases the critical Rayleigh numbers and reduces the critical wave numbers. For the case of mono-cellular convective motion ($k_c \rightarrow 0$), the maximum limit of vibration frequency is obtained as a function of $Pr^*$ and $Fr$ numbers ($\Omega_{max} \approx (\pi\, Fr\, (24\, Pr^*)^{1/2})^{-1}$). It is shown that, under high frequency and small amplitude of vibration, the response of the system is always harmonic. The results of the time-averaged method are compared with the results of the numerical method obtained in the previous study. In the framework of the assumptions for the time-averaged method, the agreement between the results is good. The present research can be used as a theoretical framework for an experimental investigation.

## Acknowledgment

This study was carried out with financial support of CNES (French National Space Agency) and San Jose State University.

The first author acknowledges the fruitful discussions he had with Professor MC. Charrier Mojtabi in 2006.

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